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NONLINEAR BEHAVIORAL BALANCING BY EXTENSION OF LIE SEMIGROUPS

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Abstract: In a previous paper Lopezlena (2004) introduced a balancing condition for nonlinear systems. This paper provides an extended justification of such balancing condition in terms of semigroups of diffeomorphisms in a Hilbert submanifold framework. Moreover, it is argued that when such condition is satisfied the resulting group of diffeomorphisms describes the flow of the nonlinear system. Using the same framework the nonlinear behavioral operator is defined and a result regarding its spectral properties is presented. *Copyright ©2006 IFAC*

Keywords: Nonlinear systems, model approximation, model reduction, differential geometric methods, geometric approaches.

1. INTRODUCTION

In the classical linear balanced reduction method (Moore, 1981), an exponentially stable linear system realization is said to be balanced when the controllability Gramian equals the observability Gramian and it is called axis-balanced when additionally both Gramians are diagonal with an ordered spectrum, (Curtain and Zwart, 1995). Whenever such system is controllable and observable, there is an infinite number of transformations that provide us with a balanced realization and furthermore a family of transformations yield the axis-balanced condition.

Balanced reduction is important for control purposes since the minimality properties of the original system are preserved in the reduced order system. Such preserved properties are clearly coordinate-invariant.

Nowadays nonlinear balanced reduction has been discussed in several publications. In (Scherpen, 1993) a nonlinear generalization of Moore's approach has been presented using energy functions keeping certain similarity with the approach of behavioral balanced reduction for linear systems due to (Weiland, 1994). In (Fujimoto and Scherpen, 2005) it is shown that the nonlinear balancing problem can be solved with the solution of a nonlinear eigenvalue problem. Nonlinear adjoint operators for this purpose are presented in (Fujimoto *et al.*, 2002). Schmidt pairs for the nonlinear balancing problem are proposed in (Gray and Scherpen, 2005).

There is a rich geometric structure behind the nonlinear extension of Moore's work. In (Lopezlena, 2004) some differential geometry-oriented research began towards an adequate geometric theory for the nonlinear extension of the behavioral balancing of (Weiland,

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1994). In this paper such geometric approach continues in development with emphasis on the framework of dissipative systems initiated in (Lopezlena *et al.*, 2003) and using Hilbert submanifold theory and semigroups of diffeomorphisms, firstly used in (Lopezlena, 2004) for nonlinear balanced reduction purposes.

The exposition is planned as follows: In Sec. 2 essential geometric concepts are introduced along with semigroup theory and the Hilbert submanifold structures. Moreover, based on a theorem due to (Mirotin, 2001), it is shown that when the balancing condition alluded in (Lopezlena, 2004) is satisfied, then both semigroups define a group of diffeomorphisms, which in turn defines the flow of the system. In Sec. 3 is shown that the storage functions of the Dissipativity theory are generating functions of the action of past and future group maps and moreover they can be chosen as metrics of two dual Riemannian spaces. In Sec. 4 the behavioral operator is briefly discussed in terms of semigroups of diffeomorphisms. Finally, assuming the balancing condition is satisfied, in Sec. 5 some eigenvalue problems for such operator are discussed.

2. BEHAVIORAL-GEOMETRIC CONCEPTS

Let us denote a dynamical system by Σ . Such system is perturbed by the environment through a set of variables called *manifest or external signals* defined on a space \mathcal{W} where such signals are supported. As the system evolves in time, these external signals define (behavioral) trajectories on a space $\mathfrak{B} \subset \mathfrak{t} \times \mathcal{W}$ called the *behavioral space*. The triad $(\mathfrak{t}, \mathcal{W}, \mathfrak{B})$ is said to define a dynamical system in the behavioral approach (Weiland, 1994). Associated to such system is a map from past external signals into future external signals $\tilde{\Gamma} : \mathfrak{B}_p \mapsto \mathfrak{B}_f$ which will be referred hereafter as the *behavioral operator*. This operator defines an exclusion law which discards trajectories outside the set of behavioral time-trajectories. Internally, such behavioral operator is built by summing up the effect of past external signals \mathfrak{B}_p on internal state-space trajectories on a manifold \mathcal{M} and reconstructing from \mathcal{M} the future external signals \mathfrak{B}_f .

2.1 Semigroups of diffeomorphisms

The behavioral operator, as an evolutionary operator, is properly defined in terms of semigroups of diffeomorphisms, briefly recalled here:

A family $\{\Phi(x, t), t \in \mathfrak{t}_f, x \in D \subset \mathcal{M}\}$ in a class of bounded operators in \mathcal{M} is called a 1-parameter semigroup if it is such that the mapping $\Phi : \mathbb{R}^1 \times D \rightarrow D$, $\Phi(t, x) = \phi^t(x)$ depends smoothly on $t \in \mathbb{R}^+$; $\phi^0(x) = x$ and $\phi^{t_2} \circ \phi^{t_1}(x) = \phi^{t_2+t_1}(x)$, being called a *strongly continuous semigroup* (or C_0 -semigroup) if $t \mapsto \phi^t(x)$ is continuous on $[0, \infty)$ for every $x \in \mathcal{M}$. The procedure of internally building the

behavioral operator takes us to consider some class of model structure, e.g. the nonlinear system Σ written as $\dot{x}(t) = f(x(t), u(t))$, $y(t) = h(x(t))$, where $x \in \mathbb{R}^n$ are local coordinates for a C^∞ state space manifold \mathcal{M} , f and h are C^∞ . The set of external variables $\mathcal{W} \approx \mathbb{R}^\omega$, $p + q \leq \omega$, includes $u \in \mathcal{U} \subset \mathbb{R}^p$ and $y \in \mathcal{Y} \subset \mathbb{R}^q$ as subsets. For piecewise constant control inputs $u(t)$, $u : \mathfrak{t} \mapsto \mathcal{U}$, the (smooth) time varying vector field $x \mapsto f(x, u(t))$ has an associated family of vector fields denoted by $\mathcal{F}_u = \{f_u : u \in \mathcal{U}\}$. The semi-trajectories of this system are continuous curves $\varrho(t)$ on \mathcal{M} on an interval $[0, T]$ that define *integral curves* of the family \mathcal{F}_u if there exists a partition $0 = t_0 \leq t_1 \leq \dots \leq t_m = T$ and associated vector fields $\xi_1, \dots, \xi_m \in \mathcal{F}_u$ such that the restriction of $\varrho(t)$ to each open interval (t_i, t_{i+1}) , $i = 0, \dots, m$, is differentiable and such that $d\varrho(t)/dt = \xi_i(\varrho(t))$, $\varrho(0) = \varrho^0$. The formal details can be reviewed in (Lopezlena, 2004) and references therein.

Since usually we are concerned with a forward-time evolution of the system, the solution of Σ is provided in terms of 1-parameter C_0 -semigroups of diffeomorphisms $x(t) = \Phi(t; t_0, x_0, u(t))$, using *appropriately defined* piecewise-constant control inputs $u(t) \in \mathcal{U}$ that guarantee boundedness of such semi-group actions. Therefore throughout this paper we assume that such inputs $u(t)$ are uniquely characterized using storage functions as in (Lopezlena, 2004).

Recall that a vectorfield $\xi(x)$ is called a *generator* of a 1-parameter group of diffeomorphisms if it is such that $\xi(x) = [\partial\Phi(t, x)/\partial t]|_{t=0}$. The *exponential map* $\exp_x : T_x\mathcal{M} \mapsto \mathcal{M}$ is a suggestive notation in Lie groups to express that $g^t(x) = \exp(t\xi)x$ is a 1-parameter group of diffeomorphisms with generator ξ , see e.g. (Olver, 1993).

A (closed) Lie semi-group S_G of a Lie Group G is generated by the images of all the 1-parameter semigroups $\phi^t(x) : \mathbb{R}^1 \mapsto S_G$, $t \mapsto \exp(t\xi)$, being called differentiable whenever each operation $\circ : S_G \times S_G \mapsto S_G$ yields a differentiable map.

We will adopt as notation to describe the conventional forward-time evolution by the interval $\mathfrak{t} = \{t | t \in \mathbb{R}^+\}$ and a backward-time evolution by $\tau = \{\tau | \tau = -t, t \in \mathfrak{t}\}$. Moreover we consider two *half-spaces* $\mathcal{M} \times \mathfrak{t}$ and $\mathcal{M}^* \times \tau$, where \mathcal{M} and \mathcal{M}^* are dual spaces joined at $t = 0$ by duality relations at their boundary or *edge* \mathcal{M}_0 and \mathcal{M}_0^* respectively. Under this notation, there can be defined a semi-trajectory $x(t) \in \mathcal{M} \times \mathfrak{t}$, $t \in \mathfrak{t}$ generated by a positive semigroup and a negative semi-trajectory $\hat{x}(\tau) \in \mathcal{M}^* \times \tau$, $\tau \in \tau$ generated by a negative semigroup.

There is no reason to believe *a priori* that the integral trajectories of a tangent vectorfield ξ are defined for both positive $t \in \mathbb{R}^+$ and negative $\tau \in \mathbb{R}^-$ evolving times. Denote by $S_{\mathcal{F}_u}^+$ the positive semigroup of all the elements of $\{\Phi^t x(t)\}$ evolving in forward-time, i.e.

$$\Phi^t(x^0) = \exp t_k \xi_k \cdots \exp t_2 \xi_2 \cdots \exp t_1 \xi_1 x^0, \quad (1)$$

such that $t_i \geq 0$ for $i = 1, \dots, k$. A vectorfield $\xi(x)$ is a generator of the Lie semigroup of diffeo-

morphisms $S_G^+ = \{\Phi^t(x)\}_{t \in \mathfrak{t}}$ if the limit $\xi(x) = \lim_{t \rightarrow 0^+} (\Phi^t(x) - x)/t$ exists in a domain $D(\xi)$. When the solution of system Σ is solved in negative time, it defines an evolution operator map (C_0 -semigroup of diffeomorphisms) defined by $\hat{x}(\tau) = \Theta(\tau; \tau_0, \hat{x}(0), \hat{u}(\tau))$, $\tau \in \tau$, $\hat{u}(\tau) \in \mathcal{U}^*$. Using appropriately defined $\hat{u}(\tau) \in \mathcal{U}^*$, the evolution of such backward-time semigroup can be expressed as

$$\Theta^\tau(\hat{x}^0) = \exp \tau_k \xi_k \cdots \exp \tau_2 \xi_2 \cdot \exp \tau_1 \xi_1 \hat{x}^0, \quad (2)$$

with $\tau_i \leq 0$ for $i = 0, \dots, k-1, k$, denoted by $S_{\mathcal{F}_u}^-$. A vectorfield $\xi(x)$ generates $S_{\mathcal{F}_u}^- = \{\Theta^\tau(\hat{x})\}_{\tau \in \tau}$ whenever $\xi(x) = \lim_{\tau \rightarrow 0^-} (\Theta^\tau(\hat{x}) - \hat{x})/\tau$ exists in a domain $D(\xi)$.

$$\begin{array}{ccccc} \mathcal{M}_f & \xleftarrow{\Phi} & \mathcal{M}_0 & \xleftarrow{\Theta^{-1*}} & \mathcal{M}_p \\ * \parallel & & * \parallel & & * \parallel \\ \mathcal{M}_f^* & \xrightarrow{\Phi^{-1*}} & \mathcal{M}_0^* & \xrightarrow{\Theta} & \mathcal{M}_p^* \end{array}$$

Consider now the set of points at the edge \mathcal{M}_0 . Since each edge-point $x_0 \in \mathcal{M}$ defines uniquely a positive semi-trajectory $x(t)$ in forward-time and analogously for $\hat{x}_0 \in \mathcal{M}^*$ in backward-time, then the *duality pairing* between such dual spaces defines the required condition to define a complete integral trajectory, namely the semi-groups $\Phi^t(x)$, $t \in \mathfrak{t}$ and the dualized $\Theta^\tau(\hat{x})$, $\tau \in \tau$ must be such that

$$\begin{cases} [\Theta^\tau(\hat{x}(\tau))]^{-1} = \Phi^t(x(t)), & \text{s.t.} \\ [\nabla_x \Theta^\tau(\hat{x}(\tau))]_{\tau=0}^{-1} = \nabla_x \Phi^t(x(t))|_{t=0}, & \text{is regular} \end{cases} \quad (3)$$

is satisfied. Let $\mathbf{G}(\mathcal{F}_u)$ denote the (connected) Lie group of diffeomorphisms in \mathcal{M} generated by the union of $\{\exp t\xi \mid t \in \mathfrak{t}, \xi \in \mathcal{F}_u\}$. When such ξ generates integral trajectories for all times (positive and negative), then $\xi(x)$ is a *complete vector field* and the family \mathcal{F}_u is complete. Such complete vectorfield generates the group $\mathbf{G}(\mathcal{F}_u) = \{\Phi^t(x)\}_{t \in \mathfrak{t}}$ and furthermore it is the *unique group extension*, see e.g. (Hofmann and Lawson, 1983), whenever $\mathbf{G}(\mathcal{F}_u)$ is connected and the tangent *wedge* of $S_{\mathcal{F}_u}$, defined as $\mathcal{L}(S_{\mathcal{F}_u}) := \{\xi \in \mathcal{L}(\mathbf{G}(\mathcal{F}_u)) \mid \exp(t\xi) \subseteq S_{\mathcal{F}_u}\}$ is a Lie-semialgebra, (Mirotin, 2001). Furthermore, such unique group extension $\mathbf{G}(\mathcal{F}_u) = \Phi^t(x(t))$, $\mathbf{G}(\mathcal{F}_u) = S_{\mathcal{F}_u}^+ \cup S_{\mathcal{F}_u}^-$, defines the flow of Σ , being the smallest local group containing $S_{\mathcal{F}_u}^+$, (Hofmann and Lawson, 1983).

Remark 2.1. Eq. (3) is valid for lumped-parameter systems. The infinite-dimensional semigroup extension for distributed parameter systems is equivalent to Eq. (3) and can be obtained based on (Mirotin, 2002).

2.2 Hilbert manifold structures

A *Riemannian Hilbert manifold* \mathcal{M} is a differentiable manifold locally modelled on a separable Hilbert

space (Palais, 1963; Lang, 1999). Being \mathcal{M} Riemannian, it has as inner product $\mathfrak{g}_{\mathcal{M}}$ for $T_x \mathcal{M}$ equivalent to the inner product $\langle \cdot, \cdot \rangle$ in $(d\mathfrak{F}, \langle \cdot, \cdot \rangle)$ for all $x \in \mathcal{M}$, where \mathfrak{F} consists of (the equivalence class of) Lebesgue-measurable, (square) integrable functions mapping the interval $[a, b]$ into \mathbb{R}^n , denoted by $\mathcal{L}_2^n[a, b]$. Four Hilbert manifold structures are needed throughout the paper. The first one characterizes the natural duality of the state and costate spaces of the compact, differentiable, manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle_{T\mathcal{M}})$ where the internal system trajectories and associated functions are supported for inner products defined by

$$\langle \xi, \zeta \rangle_{T\mathcal{M}} = (\rho_0, \varrho_0)_0 + \int_a^b \xi^i \zeta^i dt, \quad (4)$$

$$\langle \alpha, \beta \rangle_{T^*\mathcal{M}} = (f(x_0), g(x_0))_0 + \int_{x(a)}^{x(b)} \alpha_i \beta_i dx, \quad (5)$$

where $\xi, \zeta \in T\mathcal{M}$ and $\alpha = df, \beta = dg \in T^*\mathcal{M}$ for a duality pairing

$$\begin{aligned} \langle \alpha, \xi \rangle_{T^*\mathcal{M} \times T\mathcal{M}} &= (x_0, F(x_0))_0 + \int_a^b i_\xi \alpha dt \\ &= F(x_0) + F(x(t))|_a^b. \end{aligned} \quad (6)$$

A second Hilbert structure characterizes the space of external signals $(\mathcal{W}, \langle \cdot, \cdot \rangle_{T\mathcal{W}})$ and its dual. Since it is completely analog to the previous structure, it is omitted.

The third and fourth Hilbert structures are defined by duality of the halfspaces of past $\mathcal{M}^* \times \tau$ and future $\mathcal{M} \times \mathfrak{t}$ system trajectories and the past $\mathcal{W}^* \times \tau$ and future $\mathcal{W} \times \mathfrak{t}$ behavioral trajectories. Each pair of spaces can be interpreted as two intertwined dual manifolds with forward-time evolution in $\mathcal{M} \times \mathfrak{t}$ and backward-time evolution in $\mathcal{M}^* \times \tau$. The edge can always be defined by dualization via the metric.

The system trajectories $m = (x, \hat{x}) \in \mathcal{M} \oplus \mathcal{M}^*$ with tangent vectorfields $\mu = (\xi, \alpha) \in T\mathcal{M} \oplus T^*\mathcal{M}$ are on Hilbert manifolds $(\mathcal{M}^* \times \tau, \langle \cdot, \cdot \rangle_p)$ and $(\mathcal{M} \times \mathfrak{t}, \langle \cdot, \cdot \rangle_f)$ with inner products defined by

$$\langle \mu_1^p, \mu_2^p \rangle_p = \frac{1}{2} \int_0^{-T} i_{\xi_1} \alpha^2 + i_{\xi_2} \alpha^1 d\tau, \quad (7)$$

$$\langle \mu_f^1, \mu_f^2 \rangle_f = \frac{1}{2} \int_0^T i_{\xi_1} \alpha^2 + i_{\xi_2} \alpha^1 dt, \quad (8)$$

with $Z_d \in \mathcal{L}(\tilde{\mathcal{M}}, \tilde{\mathcal{M}})$ in a duality pairing defined as

$$\langle \mu_f, \mu^p \rangle_{f \times p} = \frac{1}{2} \int_{-T}^T \mu_f^T Z_d \mu^p dt. \quad (9)$$

The halfspaces of behavioral trajectories $(\mathfrak{B}_p, \langle \cdot, \cdot \rangle_{T\mathfrak{B}_p})$ and $(\mathfrak{B}_f, \langle \cdot, \cdot \rangle_{T\mathfrak{B}_f})$ have inner products defined by

$$\langle w_1^p, w_2^p \rangle_{T\mathfrak{B}_p} = \int_0^{-T} w_1^{pT} Z_p w_2^p d\tau, \quad (10)$$

$$\langle w_f^1, w_f^2 \rangle_{T\mathfrak{B}_f} = \int_0^T w_f^{1T} Z_f w_f^2 dt, \quad (11)$$

where $w_1^p, w_2^p \in T\mathfrak{B}_p$ and $w_f^1, w_f^2 \in T\mathfrak{B}_f$ for a duality pairing defined by

$$\langle w_f, w^p \rangle_{T\mathfrak{B}_f \times p} = \int_{-T}^T w_f^T Z_d(w(t)) w^p dt, \quad (12)$$

where throughout the paper we assume $Z_p, Z_f, Z_d \in \mathcal{L}(T\mathcal{W}, T\mathcal{W})$ are self-adjoint linear operators with associated quadratic function $r : T\mathcal{W} \times T\mathcal{W} \mapsto \mathbb{R}^1$ satisfying $r(w(t)) = w^T(t) Z_i w(t) \geq 0, \forall t \in \mathfrak{t}, w \in T\mathcal{W}$.

Influenced by the past $\mathfrak{B}_p \subset \mathcal{W}^* \times \tau$, the system trajectories are used to define the future behavior $\mathfrak{B}_f \subset \mathcal{W} \times \mathfrak{t}$. The Behavior and the system trajectories are related by two storage functions associated to system Σ , the *backward-time required supply*, $S_r : \mathcal{M}^* \rightarrow \mathbb{R}^+$,

$$S_r^*(\hat{x}_0, r_r) = - \sup_{\substack{v(\cdot) \in \mathcal{V} \\ x_0 = x, T \geq 0}} \int_0^{-T} r_r(w^p(\tau)) d\tau, \quad (13)$$

and the *available storage*, $S_a : \mathcal{M} \rightarrow \mathbb{R}^+$, defined by

$$S_a(x_0, r_a) = \sup_{\substack{u(\cdot) \in \mathcal{U} \\ x_0 = x, T \geq 0}} - \int_0^T r_a(w_f(t)) dt, \quad (14)$$

where the function of external signals defined by $r : \mathcal{W} \rightarrow \mathbb{R}^1, r(w(t))$, is called *supply rate* relative to S_r^* or S_a respectively.

Assumption 2.1. The functionals $S_r^*(\hat{x}^0, r_r), S_a(x^0, r_a)$ associated to state-trajectories on \mathcal{M} and \mathcal{M}^* are induced metrics of past $\mathfrak{g}_{\mathfrak{B}}^p$ and future $\mathfrak{g}_{\mathfrak{B}}^f$ behavioral trajectories preserving the following relations

$$S_r^*(\hat{x}^0, r_r) = \langle \mu^p, \mu^p \rangle_p = \langle w^p, w^p \rangle_{T\mathfrak{B}_p}, \quad (15)$$

$$S_a(x^0, r_a) = \langle \mu_f, \mu_f \rangle_f = \langle w_f, w_f \rangle_{T\mathfrak{B}_f}. \quad (16)$$

3. PROPERTIES OF THE STORAGE FUNCTIONS

Eqs. (13) and (14) are generating functions of state-costate group actions:

Proposition 3.1. Assume that $S_a : \mathbb{R}^n \mapsto \mathbb{R}^1$ exists and is smooth on (the compact) \mathcal{M} . The (maximal) flow $\{Q^t(x)\}_{t \geq 0}$ generated by the vectorfield $\xi_q = -\nabla S_a$ is a positive semigroup; meaning that for all $t \in \mathfrak{t}_f$, $Q^t(x)$ is defined on all \mathcal{M} . Moreover, for any $x \in \mathcal{M}$, $\{Q^t(x)\}_{t \geq 0}$, has at least one critical point of S_a as a limit point as $t \rightarrow \infty$.

Proof. (Sketch). The smooth vectorfield ∇S_a (dual to dS_a) points to the direction of fastest increase of S_a . Let $\{Q^t(x)\}$ denote the maximal flow generated by $\xi_q = -\nabla S_a$, then we may write $\frac{d}{dt} Q^t(x) = \xi_q(Q^t(x)) = -\nabla S_a(Q^t(x))$, thus $\frac{d}{dt} S_a(Q^t(x)) = \nabla S_a(-\nabla S_a(Q^t(x))) = -\|\nabla S_a(Q^t(x))\|^2$. The proof is completed by showing that ξ_q is of bounded length and thus is a generator, see (Palais and Terng, 1988). ■

Proposition 3.2. Assume that $S_r^* : \mathcal{M}^* \mapsto \mathbb{R}^1$ exists and is smooth on (the compact) \mathcal{M}^* . The (maximal) flow $\{P^\tau(\hat{x})\}_{\tau \leq 0}$ generated by the vectorfield $\xi_p = -\nabla S_r^*$ is a negative semigroup on $\mathcal{M}^* \times \tau$; meaning that for all $\tau \in \tau$, $P^\tau(\hat{x})$ is defined on all \mathcal{M}^* . Moreover, for any $\hat{x} \in \mathcal{M}^*$, $\{P^\tau(\hat{x})\}$ has at least one critical point of S_r^* as a limit point as $\tau \rightarrow -\infty$.

Proof. Omitted. Similar to the proof of Prop. 3.1. ■

Group extension of these semigroup actions brings about interesting consequences:

Proposition 3.3. If condition (3) for $Q^t : \mathfrak{t} \times \mathcal{M} \mapsto \mathcal{M}^*$, $Q^t(x) = -\nabla S_a(x)$ and $P^\tau : \tau \times \mathcal{M}^* \mapsto \mathcal{M}$, $P^\tau(\hat{x}) = -\nabla S_r^*(\hat{x})$, expressed by

$$\begin{cases} [P^\tau(\hat{x}(\tau))]^{-1} = Q^t(x(t)), & \text{s.t.} \\ [\nabla_{\hat{x}}^2 S_r^*(\hat{x}, r_r)]^{-1} = \nabla_x^2 S_a(x, r_a), & \text{is regular} \end{cases} \quad (17)$$

is satisfied, then:

- (1) Both semigroups $Q^t(x)$ and $P^\tau(\hat{x})$ have a common complete generator $-\xi_p = \xi_q = \xi$.
- (2) Generated by ξ , $\mathbf{G}(\mathcal{F}_u) = Q^t(x)$ is the unique Lie group extension with associated singular value problem defined by

$$\Upsilon^t(x(t)) - \sigma^2 x(t) = 0, \quad (18)$$

for $\sigma^2 \in \mathbb{R}^1$ and $x(t) \in \mathcal{M} \times \mathfrak{t}$ with the homeomorphism $\Upsilon^t : \mathfrak{t} \times \mathcal{M} \mapsto \mathcal{M}$ given by

$$\Upsilon^t(x(t)) = P^\tau \circ Q^t(x(t)). \quad (19)$$

Proof. (1) By Props. 3.1 and 3.2 there are generators ξ_p, ξ_q (possibly other than $-\xi_p$ of Θ and ξ_f of Φ). Preservation of the group extension condition (17) implies $-\xi_p = \xi_q$. Moreover item (2) follows straightforwardly from Eq. (17) since $P^\tau \circ Q^t(x(t)) = [Q^t]^{-1} \circ Q^t(x(t))$, $t \in \mathfrak{t}$ being thus a singular value problem. ■

In view of the previous results, there remains the question of whether there exist an invariant relationship preserved throughout the evolution of the system. The following result provides an answer:

Proposition 3.4. (Duality Legendre transform). The following statements are equivalent:

- (1) The functions $S_a(x(t), r)$, $x(t) \in \mathcal{M}$, $t \in \mathfrak{t}$ and $S_r^*(\hat{x}(\tau), r)$, $\hat{x}(\tau) \in \mathcal{M}^*$, $\tau \in \tau$ are such that:

$$\begin{aligned} L(x(t), \hat{x}(\tau)) &\stackrel{\text{def}}{=} S_a(x(t), r) + S_r^*(\hat{x}(\tau), r) \\ &+ \langle \xi(x), \zeta(\hat{x}) \rangle_{T^* \mathcal{M} \times T \mathcal{M}} = 0. \end{aligned} \quad (20)$$

is preserved, where $\dot{x}(t) = \xi(x(t))$ and $\dot{\hat{x}}(\tau) = \zeta(\hat{x}(\tau))$ and $\langle \xi(x), \zeta(\hat{x}) \rangle_{T^* \mathcal{M} \times T \mathcal{M}}$ is defined in Eq. (6).

- (2) Invariance of the Legendre transform (20) is equivalent to

$$\hat{x}(\tau) = -\nabla^T S_a(x(t), r), \quad (21)$$

$$x(t) = -\nabla^T S_r^*(\hat{x}(\tau), r). \quad (22)$$

(3) Assuming that $\nabla^T S_a(x(t), r)$ has a regular inverse, (21) and (22) are related by

$$[\nabla^T S_a(x(t), r)]^{-1} = \nabla^T S_r^*(\hat{x}(\tau), r). \quad (23)$$

Proof. (Outline). (1) Coordinate independence is inherited by the dual pairing of Eq. (6) and structural arguments. (2) Since $L(x, \hat{x})$ is invariant then $\nabla L(x, \hat{x}) = 0$, i.e.

$$\frac{\partial L(x, \hat{x})}{\partial [x, \hat{x}]} = \frac{\partial L(x, \hat{x})}{\partial x} \Big|_{\hat{x}=\hat{x}^0} + \frac{\partial L(x, \hat{x})}{\partial \hat{x}} \Big|_{x=x^0} = 0,$$

then using Eq. (6), $\nabla_x^T L(x, \hat{x}) = \nabla^T S_a(x, r) + \hat{x} = 0$ and $\nabla_{\hat{x}}^T L(x, \hat{x}) = \nabla^T S_r^*(\hat{x}, r) + x = 0$, which are precisely Eqs. (21) and (22). (3) Eq. (23) is another statement of Eq. (17). In this context, since $dS_r^*(\hat{x}, r) = \sum_i^n \partial_i S_r^*(\hat{x}, r) d\hat{x}^i$ and on the other side by (20), $dS_r^*(\hat{x}, r) = d[-(x, \hat{x}) - S_a(x, r)]$ and $dS_a(x, r) = \sum_i^n \partial_i S_a(x, r) dx^i$ then $dS_r^*(\hat{x}, r) = -\sum_i^n x^i d\hat{x}^i + \hat{x}^i dx^i + \partial_i S_a(x, r) dx^i = -\sum_i^n x^i d\hat{x}^i$ (due to Eq. (21)) and the last equality is precisely Eq. (22). Conclude that one transformation is the inverse of the other and thus Eq. (23) is obtained. ■

4. THE BEHAVIORAL OPERATOR

In this section, the geometric structure of the nonlinear behavioral operator $\tilde{\Gamma} : \mathfrak{B}_p \mapsto \mathfrak{B}_f$ is presented. Consider the set of input trajectories $\{u(t)|u(t) \in \mathcal{U}, t \in \mathbf{t}\}$ on the set of admissible inputs \mathcal{U} such that a point $x^0 \in \mathcal{M}$ can be reached from the origin following a trajectory $x(t)$. We assert that u_i is equivalent to u_j , $u_i \equiv u_j \pmod{x(t)}$, $u_i, u_j \in \mathcal{U}$, if both produce the same trajectory $x(t) = G(u_i) = G(u_j)$ where $G : \mathcal{U} \mapsto \mathcal{M}$ defines a (regular by assumption) equivalence relation $u_i G u_j$. Furthermore, consider the set of state trajectories $\{x(t)|x(t) \in \mathcal{M}_0, x(0) = x^0, t \in \mathbf{t}\}$ that produce an output $y(t) \in \mathcal{Y}$. We assert that x^i is equivalent to x^j , $x^i \equiv x^j \pmod{y(t)}$, $x^i, x^j \in \mathcal{M}$, if both produce the same output trajectory $y(t) = h(x^i) = h(x^j)$ where $h : \mathcal{M} \mapsto \mathcal{Y}$ defines a (regular by assumption) equivalence relation $x^i h x^j$. The following definitions are used throughout:

Definition 4.1. Denote by $h^{-1} : \mathcal{Y} \mapsto \mathcal{M}$, $x(t) = h^{-1}(y(t))$ the inverse map of h and denote by $g^{-1} : \mathcal{M} \mapsto \mathcal{U}$, $u(t) = g^{-1}(x(t))$ the inverse map of G .

4.1 Structure of the nonlinear behavioral operator

Definition 4.2. Associated to system Σ define the following operators $\Psi_p : \mathcal{L}_2[-T, 0] \mapsto \mathbb{R}^n$, $\Psi_f : \mathbb{R}^n \mapsto \mathcal{L}_2[0, T]$ by

$$\Psi_p u(t) := \Theta^{-1}(0, -T, 0, u(t)), \quad (24)$$

$$\Psi_f(x^0) := h[\Phi(T, 0, x^0, u(t))], \quad (25)$$

where $\Theta \in S_{\mathcal{F}_u}^-$ and $\Phi \in S_{\mathcal{F}_u}^+$. The composition of Eq. (24)-(25) defines the operator $\Gamma u(t) = \Psi_f \circ \Psi_p \circ u(t)$. Moreover, using Def. 4.1, the adjoint operators $\Psi_p^\dagger : \mathbb{R}^n \mapsto \mathcal{L}_2[-T, 0]$, $\Psi_f^\dagger : \mathcal{L}_2[0, T] \mapsto \mathbb{R}^n$ are defined by

$$\Psi_p^\dagger(x^0) := g^{-1}[\Theta(-T, 0, \hat{x}^0, \hat{u}(\tau))], \quad (26)$$

$$\Psi_f^\dagger \hat{y}(\tau) := \Phi^{-1*}(0, T, 0, \hat{u}(\tau); h^{-1}(\hat{y})), \quad (27)$$

where $\Phi \in S_{\mathcal{F}_u}^+$, $\Theta \in S_{\mathcal{F}_u}^-$. The composition of the Eq. (26) and (27) defines the operator $\Gamma^\dagger \hat{y}(\tau) = \Psi_p^\dagger \circ \Psi_f^\dagger \circ \hat{y}(\tau)$.

$$\begin{array}{ccccc} \mathcal{Y} & \xleftarrow{\Psi_f} & \mathcal{M}_0 & \xleftarrow{\Psi_p} & \mathcal{U} \\ * \parallel & & * \parallel & & * \parallel \\ \mathcal{Y}^* & \xrightarrow{\Psi_f^\dagger} & \mathcal{M}_0^* & \xrightarrow{\Psi_p^\dagger} & \mathcal{U}^* \end{array}$$

The Behavioral operator maps all past exogenous variables into all future exogenous variables:

Definition 4.3. (Behavioral operator). The behavioral operator $\tilde{\Gamma} : \mathcal{L}_2[-T, 0] \mapsto \mathcal{L}_2[0, T]$, $\tilde{\Gamma} : \mathfrak{B}_p \mapsto \mathfrak{B}_f$ is defined by

$$\begin{bmatrix} \hat{u}(\tau) \\ y(t) \end{bmatrix}_f = \begin{bmatrix} \Gamma^\dagger \circ \hat{y}(\tau) \\ \Gamma \circ u(t) \end{bmatrix}_p, \quad \tau \in \boldsymbol{\tau}, \quad t \in \mathbf{t} \quad (28)$$

Remark 4.1. Denote tangent maps by $\tilde{\Gamma}_* : T\mathfrak{B}_p \mapsto T\mathfrak{B}_f$. The following facts are easily verified:

- (1) Γ and Γ^\dagger satisfy $\langle \Gamma_*^\dagger y, u \rangle_{T\mathcal{W}|_u} = \langle y, \Gamma_* u \rangle_{T\mathcal{W}|_y}$, $y \in T\mathcal{W}|_y$, $u \in T\mathcal{W}|_u$ and thus are adjoint.
- (2) Each homeomorphism map $\Gamma^\dagger \circ \Gamma : \mathcal{U} \mapsto \mathcal{U}$ and $\Gamma \circ \Gamma^\dagger : \mathcal{Y} \mapsto \mathcal{Y}$ is selfadjoint.
- (3) By construction $\tilde{\Gamma}$ is an *isometric* isomorphism satisfying $\langle \xi, \zeta \rangle_{T\mathfrak{B}_p} = \langle \tilde{\Gamma}_* \xi, \tilde{\Gamma}_* \zeta \rangle_{T\mathfrak{B}_f}$.
- (4) $\tilde{\Gamma}$ satisfies $\langle \omega^p, \tilde{\Gamma}_* \omega_f \rangle_{T\mathfrak{B}_p} = \langle \tilde{\Gamma}_* \omega^p, \omega_f \rangle_{T\mathfrak{B}_f}$, $\omega^p \in T\mathfrak{B}_p$, $\omega_f \in T\mathfrak{B}_f$ and thus it is selfadjoint on \mathfrak{B} .

5. EIGENVALUE PROBLEM FOR THE BEHAVIORAL OPERATOR

Since $\tilde{\Gamma} : \mathfrak{B}_p \mapsto \mathfrak{B}_f$ is by construction an isometry for the past and future metrics we may write

$$K(\xi) = \frac{\langle (\tilde{\Gamma}^\dagger \circ \tilde{\Gamma})_* \xi, \xi \rangle_{T\mathfrak{B}_p}}{\langle \xi, \xi \rangle_{T\mathfrak{B}_p}} := \frac{II_{\mathfrak{B}}(\xi, \xi)}{I_{\mathfrak{B}}(\xi, \xi)}, \quad (29)$$

where $K(\xi)$, $\xi \in T\mathfrak{B}$ is the normal curvature of \mathfrak{B}_p , $I_{\mathfrak{B}} = \langle \xi, \xi \rangle_{T\mathfrak{B}}$ is the first fundamental form of \mathfrak{B} and $II_{\mathfrak{B}} = \langle A_\eta^{\mathfrak{B}}(\xi), \xi \rangle_{T\mathfrak{B}}$ is the second fundamental form of \mathfrak{B} with Shape operator $A_\eta^{\mathfrak{B}} : T\mathfrak{B} \mapsto T\mathfrak{B}$, $\eta \in (T\mathfrak{B})^\perp$. The eigenvalue problem of the quotient (29) consist in finding the principal directions ξ along $T\mathfrak{B}$ where $K(\xi)$ attains stationary values κ called

principal normal curvatures.

Using classical Curvature Theory the following results are obtained:

Proposition 5.1. Let $\Sigma = (\mathbf{t}, \mathcal{W}, \mathfrak{B})$ on a Hilbert submanifold $(\mathcal{V}, \langle \cdot, \cdot \rangle_{T\mathcal{V}})$ of the Hilbert manifold of external signals $(\mathcal{W}, \langle \cdot, \cdot \rangle_{T\mathcal{W}})$ s.t. $\text{supp}\mathfrak{B} = \mathcal{V} \subset \mathcal{W}$, $\dim\mathcal{V} = v$, satisfying Assumption 2.1, $I_{\mathfrak{B}} = S_r(x^0, r_r)$ and $II_{\mathfrak{B}} = S_a(x^0, r_a)$ with $A_{\eta}^{\mathfrak{B}}(\xi) = -\nabla_{\xi}^T \eta$, $\xi \in (T\mathfrak{B})$, $\eta \in (T\mathfrak{B})^{\perp}$. The following can be asserted:

- (1) A vectorfield $\zeta \in T\mathfrak{B}$, $\langle \zeta, \zeta \rangle_{T\mathfrak{B}} = 1$ is solution to the eigenvalue problem associated to $K(\zeta)$ in (29) iff ζ is an eigenvector of $A_{\eta}^{\mathfrak{B}}$.
- (2) The set of eigenvectors of $A_{\eta}^{\mathfrak{B}}$, $\{\zeta_i \mid i = 1, \dots, v; \zeta_i \in T_p\mathfrak{B}\}$, defines an orthonormal basis of $T\mathfrak{B}$.
- (3) Denote by $\mathbb{G} = [g_{ij}]$, $\mathbb{Q} = [q_{ij}]$ the metric tensors of $I_{\mathfrak{B}}$, $II_{\mathfrak{B}}$ respectively. Then $A_{\eta}^{\mathfrak{B}} = \mathbb{Q}\mathbb{G}^{-1}$, $\forall \zeta \in T_p\mathfrak{B}$ and $K(\zeta) = \det \mathbb{Q} / \det \mathbb{G}$.

Proof. (1), (2), (3) All these results can be proved using classical theory of Gaussian curvature. Due to space restrictions, these proofs are omitted. ■

Definition 5.1. (Past and Future Gramians). The past and future map homeomorphisms $P^{\tau} : \tau \times \mathcal{M}^* \mapsto \mathcal{M}$, $Q^t : \mathbf{t} \times \mathcal{M} \mapsto \mathcal{M}^*$ defined as

$$P^{\tau}(\hat{x}^0) = \Psi_p \circ \Psi_p^{\dagger}(\hat{x}^0) \quad (30)$$

$$Q^t(x^0) = \Psi_f^{\dagger} \circ \Psi_f(x^0) \quad (31)$$

are called the nonlinear *past and future Gramians*. Their composition is denoted by the map $\Upsilon : \mathbf{t} \times \mathcal{M} \mapsto \mathcal{M}$, $\Upsilon^t(x) = P^{\tau} \circ Q^t(x)$, Eq. (19).

The following result provides an eigenvalue problem associated to Eq. (30)-(31):

Theorem 5.1. Consider the nonlinear maps (30)- (31) with associated eigenvalue problem defined by Eq. (18), $\varrho(t) \in \mathcal{M}$. Then the resulting eigenvalues $\lambda = \sigma^2$ are the same eigenvalues of the operator $\Gamma^{\dagger} \circ \Gamma \circ u(t)$.

Proof. The eigenvalue problem of the behavioral operator can be stated as finding the eigenvalue $\lambda \neq 0$ and the eigenvector $0 \neq u(t) \in \mathcal{U}$ such that $\Gamma^{\dagger} \circ \Gamma \circ u(t) = \lambda u(t)$. Express such eigenvalue problem by $\Gamma^{\dagger} \circ \Gamma \circ u(t) = \Psi_p^{\dagger} \circ \Psi_f^{\dagger} \circ \Psi_f \circ \Psi_p \circ u(t) = \lambda u(t)$ which after being mapped by the nonlinear map $\Psi_p : \mathcal{U} \mapsto \mathcal{M}$, $\varrho(t) = \Psi_p(u)$ for some state trajectory $\varrho(t) \in \mathcal{M}$, yields $\Gamma^{\dagger} \circ \Gamma \circ u(t) = \Psi_p \circ \Psi_p^{\dagger} \circ \Psi_f^{\dagger} \circ \Psi_f \circ \Psi_p \circ u(t) = P^{\tau} \circ Q^t \circ \varrho(t) = \lambda \varrho(t)$ where $P^{\tau} \circ Q^t$ and $Q^t \circ \varrho(t)$ are defined from (30) and (31) yielding $P^{\tau} \circ Q^t \circ \varrho(t) - \lambda \varrho(t) = 0$.

Assume now the eigenvalue $\lambda \neq 0$ and the state trajectory (eigenvector) $0 \neq \varrho(t) \in \mathcal{M}$ are solution of $P^{\tau} \circ Q^t \circ \varrho(t) - \lambda \varrho(t) = 0$. Map this latter equation

by $\Psi_p^{\dagger} \circ Q^t : \mathbf{t} \times \mathcal{M} \mapsto \mathcal{U}$, $u := \Psi_p^{\dagger} \circ Q^t \circ \varrho(t)$, yielding $\Gamma^{\dagger} \circ \Gamma \circ u(t) - \lambda u(t) = 0$. ■

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